Partial Gradings of Algebras

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joint work with Eliezer Batista and Joost Vercruysse

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Outline of the talk

1. Hopf Algebras
2. Partial Actions of Hopf algebras
3. Partial G-gradings of the matrix algebra
4. Partial Representations and an application to partial $\mathbb{Z}_2$-gradings.

Hopf algebras

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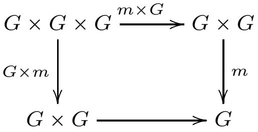
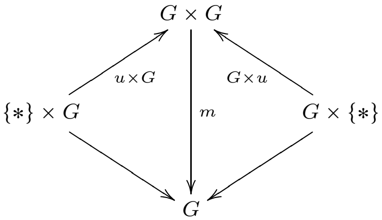


Neutral Element:



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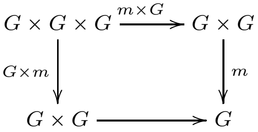

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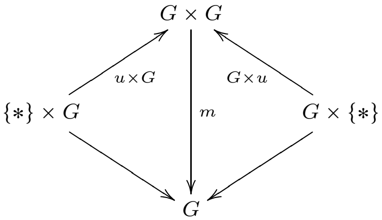

These two diagrams describe a **monoid**.

Hopf algebras

The diagram for the inverse needs more information:

one defines a new operation

$$\begin{array}{rccc} S: & G & \to & G \\ & g & \mapsto & S(g) = g^{-1} \end{array}$$

and in order to write $S(g)g = e$ as a diagram, we may consider the sequences

$$g \mapsto (g,g) \mapsto (S(g),g) \mapsto S(g)g$$

and

$$g \mapsto * \mapsto e$$

which can be written using m, u and the maps

$$\begin{array}{rccc} \Delta: & G & \to & G \times G \\ & g & \mapsto & (g,g) \end{array} \qquad \begin{array}{rccc} \varepsilon: & G & \to & \{*\} \\ & g & \mapsto & * \end{array}$$

Hopf algebras

We remark that the maps

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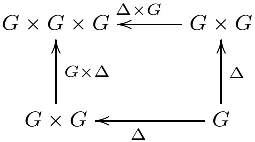
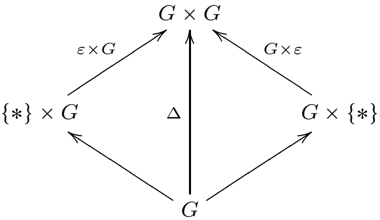
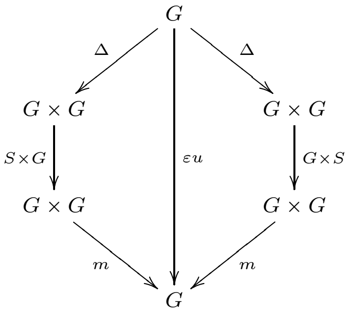
satisfy the dual diagrams (arrows reversed) of m and u respectively:





Hopf algebras

The axiom of the inverse corresponds to the following diagram:



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Let k be a field.

A **bialgebra** is a vector space B with two pairs of maps:

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A **bialgebra** is a vector space B with two pairs of maps:

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2. The comultiplication $\Delta : B \to B \otimes B$ and the counit $\varepsilon : B \to k$, satisfying the dual diagrams (B is a *coalgebra*)
(in the case of groups, there is only one possible pair (Δ, ε)).

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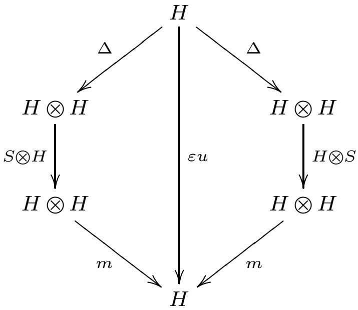
   (in the case of groups, there is only one possible pair (Δ, ε)).

These dual structures must be compatible in the sense that Δ and ε are algebra maps.

Hopf algebras

A bialgebra is a **Hopf algebra** if it has an algebra antimorphism $S : H \to H$, called the **antipode**, such that the diagram below commutes:



One can say that a Hopf algebra is a group in the category of vector spaces (and a group is a Hopf algebra in the category of sets).

Hopf algebras

Hopf algebras act on algebras, just as groups do.

Let H be a Hopf algebra, let A be an algebra. A **left action** of H on A is a structure of left H-module on A which also satisfies:

(i) $h \cdot 1_A = \varepsilon(h) 1_A$

(ii) $h \cdot (ab) = \sum (h_{(i,1)} \cdot a)(h_{(i,2)} \cdot b)$,

where $\Delta(h) = \sum h_{(i,1)} \otimes h_{(i,2)}$. One also says that A is a (left) **H-module algebra**.

Hopf algebras

There are other points in common with groups and their representations.

- The tensor product of two left H-modules M, N is again a left H-module by

$$h \cdot (m \otimes n) = \sum_i (h_{(i,1)} \cdot m) \otimes (h_{(i,1)} \cdot n)$$

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- When H acts on A, the vector space $A \otimes H$ has a structure of k-algebra given by

$$(a \otimes h)(b \otimes k) = \sum_i a(h_{(i,1)} \cdot b) \otimes h_{(i,2)} k.$$

  This algebra is called the **smash product** of A by H, and is denoted by $A \# H$.

Hopf algebras associated to groups

Let G be a *finite* group.

- The **dual group algebra** is the dual vector space k^G with the following structure:

  If $\{p_g; g \in G\}$ is the dual basis associated to G, then

  $$p_g p_h = \delta_{g,h} p_g, \quad \sum_g p_g = 1,$$

  and

  $$\Delta(p_g) = \sum_{xy=g} p_x \otimes p_y, \quad \varepsilon(p_g) = \delta_{g,e}, \quad S(p_g) = p_{g^{-1}}.$$

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- A left action of $H = k^G$ on A is a G-**grading** of A:

  $$p_g \cdot a = a_g, \quad \text{the } g\text{-component of } a \in A.$$

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- The left modules over $A\#k^G$ coincide with the G-graded left A-modules.

Hopf algebras associated to groups

Let G be a group.

- The **group algebra** kG is a Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.$$

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- A left action of $H = kG$ on A is an **action of** G **by automorphisms**.
- The smash product $A\#kG$ coincides with the skew group algebra AG.

Partial Actions of a Hopf algebra

(Caenepeel, Janssen 2006) A a unital algebra, H Hopf algebra. A (left) **partial action** of H on A, or a structure of **partial H-module algebra** on A, is a linear function $\alpha : H \otimes A \to A$, $\alpha(h \otimes a) = h \cdot a$, such that

(i) $1 \cdot a = a$,

(ii) $h \cdot (ab) = (h_{(i,1)} \cdot a)(h_{(i,2)} \cdot b)$,

(iii) $h \cdot (k \cdot a) = (h_{(i,1)} \cdot 1_A)(h_{(i,2)} k \cdot a)$

The partial action is **symmetric** if also

(iii) $h \cdot (k \cdot a) = (h_{(i,1)} k \cdot a)(h_{(i,2)} \cdot 1_A)$.

Partial Actions of a Hopf algebra

Recall that if one has a partial action of a group G on A, then the A-module $\oplus_{g\in G} D_g$ has the structure of a G-graded algebra, the **partial skew group algebra** $A *_\alpha G$.

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If A is a left H-module algebra then $A \otimes H$ is an associative algebra with the multiplication

$$(a \otimes h)(b \otimes k) = \sum_i a(h_{(i,1)} \cdot b) \otimes h_{(i,2)} k)$$

and the subalgebra $\underline{A\#H} = (A \otimes H)(1_A \otimes 1_H)$ is a unital associative algebra.

Partial Actions of a Hopf algebra

Let G be a group, A a k-algebra,

$$\alpha = (\{D_g\}_{g\in G}, \{\alpha_g : D_{g^{-1}\to D_g}\}_{g\in G})$$

a partial G-action where $D_g = 1_g A$ for all $g \in G$ (1_g central idempotent). For each $g \in G$, define an endomorphism of A by

$$\begin{array}{ccccc} A & \longrightarrow & D_{g^{-1}} & \longrightarrow & D_g \\ a & \longmapsto & a1_{g^{-1}} & \longmapsto & \alpha_g(a1_{g^{-1}}) \end{array}$$

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If $g \cdot a = \alpha_g(a1_{g^{-1}})$, then $g \cdot 1_A = 1_g$ and

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Partial actions of kG

Symmetric partial actions of kG on a k-algebra A correspond to partial actions of G on A where every ideal D_g is generated by a central idempotent.

Partial G-gradings

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  right partial H-coactions $\Leftrightarrow$ left partial H^*-actions
- On the other hand, right kG-coactions $\Leftrightarrow$ G-gradings.

Partial G-gradings of the base field

From now on, we assume that $|G| \neq 0$ in k.

- Partial G-grading of k: map $p_g \to \lambda_g \in k$.
- Define the **support** of $\lambda = (\lambda_g)_{g \in G}$ by

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Then H is a subgroup of G and

$$\lambda_g = \frac{1}{|H|}\delta_{gH,H}.$$

The partial G-gradings of k are in correspondence with the subgroups of G.

G/H-gradings and partial G-gradings

Lifting formula for G/H-gradings

Let G be a finite group, let H be a normal subgroup of G, and let A be a G/H-graded algebra. Then A is also a partial G-graded algebra by

$$p_g \cdot a = \frac{1}{|H|}\, a_{gH}\,.$$

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Note that 1_A is an eigenvector for each element of k^G, since

$$p_g \cdot 1_A = \frac{1}{|H|}\delta_{gH,H} 1_A.$$

G/H-gradings and partial G-gradings

Conversely, we will say that a partial G-grading of an algebra A **has linear support** if the unit 1_A is an eigenvector for each element of k^G.

The **linear support** is the support H of the associated partial G-grading of $k \simeq k\, 1_A$.

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Partial G-gradings and G/H-gradings

[_,Batista,Vercruysse 2013] Let G be a finite group, let H be a normal subgroup of G, let A be a k-algebra.

Every linear partial G-grading of A with support H corresponds to a G/H-grading of A by the formula

$$a_{gH} = |H| p_g \cdot a$$

and the lifting of this G/H-grading is the original partial G-grading of A.

Good Partial G-gradings of Matrix Algebras

[Dăscălescu, Ion, Năstăsescu, Rios Montes 1999]

Given $(g_1, g_2, \ldots, g_n) \in G^n$, the formula

$$\deg(E_{i,j}) = g_i g_j^{-1},\ 1 \leqslant i, j \leqslant n \tag{1}$$

defines a G-grading on $M_n(k)$.

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A G-grading of $M_n(k)$ in which every elementary matrix $E_{i,j}$ is homogeneous is called a **good grading** (also an *elementary grading* in [Bahturin, Sehgal, Zaicev 2001].

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Theorem

[DINM, 1999] Good G-gradings of $M_n(k)$ are in correspondence with elements of G^{n-1} via equation (1) above.

Good Partial G-gradings of Matrix Algebras

A partial G-grading of $M_n(k)$ is a **good partial grading** if the elementary matrices $\{E_{i,j}; 1 \leqslant i, j \leqslant n\}$ are simultaneous eigenvectors for all operators $p_g \cdot \_$.

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Characterization of good partial gradings

[_,Batista,Vercruysse 2013] Let G be a finite abelian group.

1. Every good partial G-grading of $M_n(k)$ is a linear grading.
2. Fix a subgroup H of G.
   There is a bijective correspondence between good partial G-gradings of $M_n(k)$ with linear support H and good G/H-gradings of $M_n(k)$.

Partial Representations of a group

Let G be a group, A be a k-algebra.

A **Partial Representation** of G on a algebra B is a map $\pi : G \to B$ such that

(PR1) $\pi(e) = 1_B,$

(PR2) $\pi(g)\,\pi(h)\,\pi(h^{-1}) = \pi(gh)\,\pi(h^{-1}), \qquad \forall g, h \in G$

(PR3) $\pi(g^{-1})\,\pi(g)\,\pi(h) = \pi(g^{-1})\,\pi(gh)\,, \qquad \forall g, h \in G.$

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- $\alpha_g : D_{g^{-1}} \to D_g$ extends to the endomorphism $\theta_g : A \to A$ given by $\theta_g(a) = \alpha_g(a 1_{g^{-1}})$. The map

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- Consider the k-algebra generated by the symbols $[g]$, $g \in G$, subject to the relations
  1. $[e] = 1$
  2. $[g][h][h^{-1}] = [gh][h^{-1}$
  3. $[g^{-1}][g][h] = [g^{-1}][gh]$

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  partial representations of G on $B \iff$ representations of G on B.

Partial Representations of a Hopf algebra

- Let H be a Hopf algebra, V be a vector space.

$$\begin{aligned} \text{representation of } H \text{ on } V \quad &\Leftrightarrow \quad \text{structure of } H\text{-module on } V \\ &\Leftrightarrow \quad \text{algebra map } \pi : H \to \mathrm{End}(V) \end{aligned}$$

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- (obviously) If A is an H**-module algebra** then $\pi : H \to \mathrm{End}(A)$, $\pi(h)(a) = h \cdot a$, is a **representation**.

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...what is a *partial* representation of H, i.e., what is a *partial H-module*?

Partial Representations of a Hopf algebra

[_,Batista,Vercruysse 2013] Let H be a Hopf algebra over k, B a unital k-algebra. A **partial representation** of H in B is a k-linear map $\pi : H \to B$ satisfying the conditions below.

(PR1) $\pi(1_H) = 1_B$

(PR2.1) $\pi(h)\pi(k_{(i,1)})\pi(S(k_{(i,2)})) = \pi(hk_{(i,1)})\pi(S(k_{(i,2)}))$

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Partial Representations of a Hopf algebra

If H is cocommutative, (PR1)-(PR5) are reduced to

(PR1') $\pi(1_H) = 1_B$

(PR2') $\pi(h)\pi(k_{(i,1)})\pi(S(k_{(i,2)})) = \pi(hk_{(i,1)})\pi(S(k_{(i,2)}))$

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Partial Representations of a Hopf algebra

Let A be a partial H-module algebra.

The following are partial representations of H:

$$\begin{array}{rcl} \pi : H & \longrightarrow & \mathrm{End}(A) \\ h & \mapsto & h \cdot \_ \end{array}$$

$$\begin{array}{rcl} \pi : H & \longrightarrow & \underline{A \# H} \\ h & \mapsto & 1 \# h \end{array}$$

H_{par}

Let H be a k-Hopf algebra. The "partial Hopf algebra" H_{par} is the k-algebra generated by symbols $[h]$, with $h \in H$, satisfying

$$[\alpha h + \beta k] = \alpha[h] + \beta[k], \text{ for all } \alpha, \beta \in k \text{ and } h, k \in H$$

and

(1) $[1_H] = 1_{H_{par}}$

(2.1) $[h][k_{(i,1)}][S(k_{(i,2)})] = [hk_{(i,1)}][S(k_{(i,2)})]$

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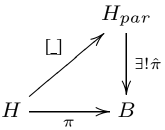
H_{par}

The linear map $h \in H \mapsto [h] \in H_{par}$ is a partial representation of H in H_{par}.

Characterization of H_{par}

The pair $(H_{par}, [\_])$ is determined by the following universal property: For every partial representation $\pi : H \to B$ there is a unique algebra morphism $\hat{\pi} : H_{par} \to B$ such that $\pi = \hat{\pi} \circ [\_]$.



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When $H = kG$ and G is a finite group, one has $H_{par} = k_{par}G$. It is well-known that

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On the other hand, it may be infinite dimensional even if H is finite dimensional (for instance, the Sweedler algebra H_4).
H_{par} is also a Hopf algebroid, and via this Hopf algebroid we can provide a monoidal structure to the category of left H_{par}-modules (i.e., we can tensor partial representations).

Theorem

Partial H-module algebras coincide with algebras in the category of left H_{par}-modules.

Partial $\mathbb{Z}_2$-gradings

- For $H = k^{\mathbb{Z}_2}$, one has the algebra isomorphism

$$\begin{array}{ccc} H_{par} & \to & \dfrac{k[x]}{\langle p(x)\rangle} \\ [p_{\overline{1}}] & \mapsto & x + \langle p(x)\rangle \end{array}$$

  where $p(x) = x(x-1)(2x-1)$.

- It follows that $T \in \mathrm{End}(V)$ defines a partial representation of $H = k^{\mathbb{Z}_2}$ in $\mathrm{End}(V)$ by $\pi(p_{\overline{1}}) = T$ if and only if its minimal polynomial divides $p(x) = x(2x-1)(x-1)$.

Partial $\mathbb{Z}_2$-gradings

Classification of Partial $\mathbb{Z}_2$-gradings

Let $H = k^{\mathbb{Z}_2}$. Let A be a partially $\mathbb{Z}_2$-graded k-algebra.

Let $A = A_0 \oplus A_1 \oplus A_{1/2}$ be its decomposition as left H_{par}-module.

(1) $B = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded subalgebra of A, with homogeneous components $B_{\overline{0}} = A_0$ and $B_{\overline{1}} = A_1$.

(2) $A_{1/2}$ is a unital ideal of A, $p_{\overline{1}} \cdot x = \dfrac{x}{2}$ for all $x \in A_{1/2}$, and

$$A \simeq B \times A_{1/2}$$

as an algebra.

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Conversely, if B is a $\mathbb{Z}_2$-graded algebra and C is a k-algebra, $B \times C$ is partially $\mathbb{Z}_2$-graded by defining

$$p_{\overline{0}} \cdot (b, c) = (b_{\overline{0}}, \frac{x}{2}), \quad p_{\overline{1}} \cdot (b, c) = (b_{\overline{1}}, \frac{x}{2}).$$

- M.M.S. Alves, Eliezer Batista, Joost Vercruysse, *Partial Representations of Hopf Algebras*, arXiv:1309.1659.
- E.R. Alvares, M.M.S. Alves, E. Batista, *Partial Hopf module categories* Journal of Pure and Applied Algebra 217 (2013) 1517-1534.
- M.M.S. Alves, E. Batista, *Enveloping Actions for Partial Hopf Actions*, Comm. Algebra 38 (2010), 2872-2902.
- S. Caenepeel, K. Janssen, *Partial (co)actions of Hopf algebras and partial Hopf-Galois theory*, Comm. Algebra 36 (2008), 2923-2946.
- S. Dăscălescu, B. Ion, C. Năstăsescu and J. Rios Montes, *Group Gradings on Full Matrix Rings*, J. Algebra 200 (1999), 709-728.
- M. Dokuchaev, R. Exel, *Associativity of Crossed Products by Partial Actions, Enveloping Actions and Partial Representations*, Trans. Amer. Math. Soc. 357 (2005) 1931-1952.
- M. Dokuchaev, R. Exel, P. Piccione, *Partial Representations and Partial Group Algebras*, J. Algebra 226 (2000) 505-532.